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Research Article

The Novel Study on the Benjamin–Bona–Mahony Equation with Conformable Fractional Operator

Hakkı GÜNGÖR[a]

[a] *Department of Computer Technology Vocational School, Ufuk University, Ankara, TURKEY*
** Corresponding author's e-mail address: hakki.gungor@ufuk.edu.tr*
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ABSTRACT

In this study, the conformable fractional Benjamin–Bona–Mahony (BBM) equation is solved by using the conformable q-formable homotopy analysis transform method (Cq-FHATM). This paper presents a practical implementation that demonstrates the possible advantages and effectiveness of the stated approach. In addition, an error analysis is undertaken to validate the scheme's precision. The accuracy of the method in the study is ensured by computational simulations. This study shows the findings obtained from numerical and graphical analysis. It also explains a computationally accurate and simple strategy to understand and solve complex phenomena in science and technology using compatible fractional nonlinear partial differential equations.

Keywords: *Benjamin-Bona-Mahony equation, formable transform, conformable fractional derivative*

Uyumlu Kesirli Operatör ile Benjamin-Bona-Mahony Denklemi Üzerine Yeni Bir Çalışma

ÖZ

Bu çalışmada, uyumlu kesirli Benjamin-Bona-Mahony (BBM) denklemi, uyumlu q-oluşturulabilir homotopi analiz dönüşümü yöntemi (Cq-FHATM) kullanılarak çözülmüştür. Bu makale, önerilen tekniğin potansiyel faydalarını ve etkinliğini gösteren bir uygulama sunmaktadır. Ek olarak, şemanın hassasiyetini doğrulamak için bir hata analizi yapılmıştır. Çalışmadaki yöntemin doğruluğu, hesaplamalı simülasyonlarla sağlanmıştır. Bu çalışma, sayısal ve grafiksel analizden elde edilen bulguları göstermektedir. Ayrıca, uyumlu kesirli doğrusal olmayan kısmi diferansiyel denklemleri kullanarak bilim ve teknolojideki karmaşık fenomenleri anlamak ve çözmek için hesaplamalı olarak doğru ve basit bir strateji açıklamaktadır.

Anahtar Kelimeler: *Benjamin-Bona-Mahony denklemi, şekillendirilebilir dönüşüm, uyumlu kesirli türev,*

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I. INTRODUCTION

Fractional partial differential equations (FPDE) are a type of differential equation in which the derivatives have both partial and fractional order. These equations are structured using fractional degrees of derivatives rather than standard integer degree derivatives. It is notably useful for modeling many natural events more precisely and realistically.

More accurate modeling of complicated or unpredictable events is possible using FPDEs. Fields including diffusion, Schrödinger equation, material science, heat transfer, fluids, chemical analysis of liquids, electromagnetic theory, mechanics, physics, control theory, and biology, as well as biophysics and bioengineering, use this fascinating subject in their work. A plethora of intriguing new applications appear daily. When fractal functions are examined in the context of studying non-linear and non-equilibrium processes, standard mathematics is recognized to be insufficient. Further applications of this technique include discovering series sums, taking some definite integrals, and fractional derivatives and integrals. Complex dynamics displayed by complex systems are described using a particular kind of mathematical model called fractional partial differential equations (FPDEs). When equation solutions are not proportionate to the input, the system is said to be non-linear, which makes it challenging to forecast how the system will behave [1–6].

Computational advancements have recently presented conformal derivatives as a substitute for conventional fractional derivatives. These derivatives offer various advantages to practitioners and researchers in terms of accessibility and simplicity of implementation, which are less available in fractional derivative notions. In cases when obtaining the precise solution is challenging, conformal derivatives offer an approximation of fractional computational operations. The ability to approximate computations increases the efficiency and practicality of techniques for solving fractional differential equations. These features include linearity, Leibniz's rule, and the composition rule, which are essential for developing analytical solutions and conducting theoretical investigations [19, 20].

An inherent challenge in addressing FPDEs is the intricate nature and absence of universally applicable remedies. The applicability of analytical approximations is modest due to their inherent limitation to specific situations. By comparison, numerical approximations offer a wider range of usefulness but necessitate substantial computational resources and meticulous verification procedures to guarantee precision and stability Despite its inherent problems, fractional nonlinear partial differential equations (PDEs) are widely used in a variety of scientific and engineering disciplines. They are particularly useful for representing and analyzing fluid dynamics, materials science, and biological systems [7-13].

Nevertheless, the fractional order exhibits temporal and spatial variability. The given scenario gives rise to a burgeoning domain of FPDEs featuring variable order fractional operators, as evidenced. A variety of robust numerical methodologies have been developed and documented in the scientific literature, including significant contributions from respected experts in the field. A number of strategies such as Adomian decomposition method, differential transform method, homotopy perturbation method, homotopy analysis method, collocation method, Sumudu transform method, Crank-Nicolson method, conformable fractional Elzaki decomposition method, conformable Shehu transform decomposition method et al. have been proposed in the literature for solving mathematical problems. Recently, there are many studies in the literature about conformable partial differential equations with different methods [14-17].

The formable integral transform, a new and efficient technique for solving ordinary and partial differential equations, has been introduced. A new method, the formable transform decomposition method (FTDM), that combines the formable integral transform and the decomposition method, has been developed. A new methodology, known as the q-homotopy analysis formable transform technique (q-HAFTT), has been developed to achieve an approximate analytical solution to the time-fractional Rosenau-Hyman problem [18].

The fractional Benjamin-Bona-Mahony (FBBM) equation is a generalized version of the classical Benjamin-Bona-Mahony equation that incorporates fractional derivatives.

The FBBM equation can be written in the form:

$$D^{\alpha}u_t(x,t) + \delta u(x,t)u_x(x,t) + \beta u_{xxt}(x,t) + \gamma u_x(x,t) = 0 \quad (1)$$

where u(x,t) represents the wave amplitude, δ, β, γ are real coefficients, α represents the order of the fractional derivative operator.

Our approach focuses specifically on the conformable time-fractional Benjamin-Bona-Mahony equation:

$$D^{\alpha}u_t(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) + u_x(x,t) = 0 \quad (2)$$

The Comformable Fractional Benjamin-Bona-Mahony (CFBBM) equation is an expansion of the traditional Benjamin-Bona-Mahony (BBM) equation, incorporating fractional derivatives that are consistent with the classical equation. The CFBBM equation improves on the classical BBM equation by introducing conformable fractional derivatives. Conformable fractional calculus research focuses on equations like the CFBBM equation, which can accurately describe physical events in domains such as fluid dynamics and nonlinear waves [20].

The novelty of this study is that the numerical solutions of the compatible fractional Benjamin-Bona-Mahony equation are obtained with a new method, Cq-FHATM and it has contributed to the literature by obtaining the numerical solutions of the compatible fractional Benjamin-Bona-Mahony equation for the first time. The utilization of the Benjamin-Bona-Mahony equation enables researchers to simulate these occurrences with greater realism. Cq-FHATM generates h-curves that demonstrate the convergence interval of the series solution obtained. The main objective of this study is to present a novel hybrid approach that combines the q-homotopy analysis transformation method and conformal conformal integral transformation: the conformal q-conformal homotopy analysis transformation method (Cq-FHATM). This method's benefit is that it allows for the adaptation of two potent computational approaches for the study of fractional differential equations (FDEs). By carefully altering the value of $\hbar$, it is possible to regulate the convergence region of the solution series within a considerable allowable domain.

The conformable derivative provides a bridge between classical derivatives and fractional derivatives, offering a simpler and more intuitive framework for fractional calculus. This derivative is particularly advantageous due to its ease of computation and ability to handle physical and engineering problems effectively. In this study, the conformable derivative is utilized to model and solve the fractional Benjamin–Bona–Mahony (BBM) equation, enabling a computationally efficient and accurate approach to addressing nonlinear partial differential equations [24].

The conformable derivative offers several significant advantages in the study and application of fractional calculus. Unlike traditional fractional derivatives such as the Riemann-Liouville and Caputo derivatives, the conformable derivative is defined in a simpler and more intuitive manner, allowing for easier implementation in analytical and numerical computations. This derivative satisfies many properties of classical calculus, including the product, chain, and quotient rules, which are not generally satisfied by other fractional derivatives. Additionally, the conformable derivative is particularly effective in modeling and solving physical and engineering problems due to its straightforward formulation and reduced computational complexity. These characteristics make it an ideal choice for addressing nonlinear fractional partial differential equations, as demonstrated in this study [27].

The subsequent section of the investigation is presented in further depth. Section 2 of the document presents the fundamental definitions of fractional derivatives and introduces the conformable fractional Formable transform (CFFT) as a means to compute these derivatives. Section 3 presents the conformable

q- Formable homotopy analysis transform method (Cq-FHATM). The numerical solutions of the time-fractional Benjamin-Bona-Mahony equation are reported in Section 4. Section 5 presents the final remarks or findings.

II. MAIN DEFINITIONS AND THEOREMS

The fundamental definitions and theorems will be covered in this section.

Definition 2.1 [27-28] Give a function $\delta: [0, \infty) \to \mathbb{R}$. The "conformable fractional derivative" (CFD) of δ order α is defined as follows

$$T_\alpha(\delta)(x) = \lim_{\mu \to 0} \frac{\delta(x + \mu x^{1-\alpha}) - \delta(x)}{\mu} \tag{3}$$

for $\forall x \in (0, \infty), \alpha \in (0, 1]$.

Lemma 2.1 [27-28] Suppose that $\alpha \in (0, 1]$ and δ, ϱ be α −differentiable at the point $x > 0$. Then, the following properties exist:

$$\boldsymbol{i}. T_\alpha(\varsigma\delta + \upsilon\varrho) = \varsigma T_\alpha(\delta) + \upsilon T_\alpha(\varrho), \text{ for } \forall \varsigma, \upsilon \in \mathbb{R} \tag{4}$$

$$\boldsymbol{ii}. T_\alpha(\delta^\rho) = p\delta^{\rho-1} \text{ for } \forall \rho \in \mathbb{R} \tag{5}$$

$$\boldsymbol{iii}. T_\alpha(\tau) = 0, \text{ for } \forall \text{ constant } \tau \tag{6}$$

$$\boldsymbol{iv}. T_\alpha(\delta\varrho) = \delta T_\alpha(\varrho) + \varrho T_\alpha(\delta) \tag{7}$$

$$\boldsymbol{v}. T_\alpha\left(\frac{\delta}{\varrho}\right) = \frac{\varrho T_\alpha(\delta) - \delta T_\alpha(\varrho)}{\varrho^2} \tag{8}$$

Definition 2.2 [33] Suppose that $\varrho: [0,\infty) \to \mathbb{R}$ is a function. FT of the function ϱ is described as follows:

$$R[\varrho(t)] = B(s; u) = \frac{s}{u} \int_0^\infty exp\left(\frac{-st}{u}\right) \varrho(t) dt \tag{9}$$

Definition 2.3 Suppose that $\alpha \in (0, 1]$, $\varrho: [0,\infty) \to \mathbb{R}$ is a function. CFFT of the function ϱ is described as follows:

$${}_cR_\mu[\varrho(t)] = B_\alpha(s; u) = \frac{s}{u} \int_0^\infty exp\left(\frac{-st^\alpha}{u\alpha}\right) \varrho(t) t^{\alpha-1} dt \tag{10}$$

Definition 2.4 Suppose that $0 < \alpha \leq 1$, $\varrho: [0,\infty) \to \mathbb{R}$ is real function. The CFFT for the CFD of the function $\varrho(t)$ is defined as

$$B_\alpha[T_\alpha \varrho(t)](u) = \frac{s}{u} B_\alpha(s; u) - \frac{s}{u} \varrho(0) \tag{11}$$

III. ANALYSIS OF THE Cq-FHATM

Now, analyze the conformable time FPDEs (CTFPDEs):

$$D_t^{\alpha}u(x,t) + Ku(x,t) + Su(x,t) = \zeta(x,t) \tag{12}$$

for all $t > 0,\ n-1 < \alpha \leq n$.

where K and S are linear and nonlinear operators, $\zeta(x,t)$ is a nonhomogeneous function, and D_t^{α} is CFD of order α.

Step 1. By applying the CFFT to Equation (12) and utilizing the starting condition (IC), Equation (13) can be determined by

$$\frac{s}{u}B_{\alpha}(s;u) - \frac{s}{u}g(0) + {}_cR_{\alpha}[Ku(x,t)] + {}_cR_{\alpha}[Su(x,t)] = {}_cR_{\alpha}[\zeta(x,t)]. \tag{13}$$

Rewriting the Eq. (13), Eq. (14) is obtained by

$${}_cR_{\alpha}[u(x,t)] - u(x,0) + \frac{u}{s}\ {}_cR_{\alpha}[Ku(x,t)] + \frac{u}{s}\ {}_cR_{\alpha}[Su(x,t)] - \frac{u}{s}\ {}_cR_{\alpha}[\zeta(x,t)] = 0 \tag{14}$$

Step 2. Via the homotopy analysis method, the nonlinear operator of u ⃗(x,t;q) is described by

$$N^1[\vec{u}(x,t;q)\ ] = {}_cR_{\alpha}[\vec{u}(x,t;q)\ ] - \vec{u}(x,t;q)\ (0^+) + \frac{u}{s}(\ {}_cR_{\alpha}[K\vec{u}(x,t;q)] + {}_cR_{\alpha}[S\vec{u}(x,t;q)] - {}_cR_{\alpha}[\zeta(x,t)]) \tag{15}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

A homotopy is generated by

$$(1-nq)\ {}_cR_{\alpha}[\vec{u}(x,t;q) - u_0(x,t)] = \hbar q H^*(x,t)\ {}_cR_{\alpha}[\vec{u}(x,t;q)] \tag{16}$$

where ${}_cR_{\alpha}$ signifies CFFT and $\hbar \neq 0$ is an auxiliary parameter. The outcomes of Equation (16) are determined for $q = 0$ and $q = \frac{1}{n}$ by

$$\vec{u}(x,t;0) = u_0(x,t), \vec{u}\left(x,t;\frac{1}{n}\right) = u(x,t). \tag{17}$$

With q increasing from 0 to $1/n$, $\vec{u}(x,t;q)$ converges to the solution $u(x,t)$ from $u_0(x,t)$. When the Taylor theorem is applied to q, the result is found by

$$\vec{u}(x,t;q) = u_0(x,t) + \sum_{i=1}^{\infty} u_m(x,t)q^m \tag{18}$$

where,

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \vec{u}(x,t;q)}{\partial q^m}|_{q=0}. \tag{19}$$

for $u_0(x,t)$, n and $\hbar$, which are convenient, Equation (18) converges at q $= \frac{1}{n}$.

By dividing by $m!$ and differentiating the $0-th$ order deformation Equation (18) m times with regard to q, one obtains for $q = 0$.

$$ {}_cR_\alpha[u_m(x,t) - \kappa_m u_{m-1}(x,t)] = \hbar H^{1*}(x,t)\mathcal{R}_{1,m}(\vec{u}_{m-1}) \tag{20}$$

Step 3. When inverse CFFT (ICFFT) is used on Equation (20), the result is acquired as

$$u_m(x,t) = \kappa_m u_{m-1}(x,t) + \hbar\left({}_cR_\alpha\right)^{-1}\left[H^{1*}(x,t)\mathcal{R}_{1,m}(\vec{u}_{m-1})\right] \tag{21}$$

where

$$\mathcal{R}_{1,m}(\vec{u}_{m-1}) = {}_cR_\alpha[u_{m-1}(x,t)] - \left(1-\frac{\kappa_m}{n}\right)u_0(x,t) + \frac{u}{s}\ {}_cR_\alpha[Ku_{m-1}(x,t) + {H^{1*}}_{m-1}(x,t) - \zeta(x,t)] \tag{22}$$

and

$$\kappa_m = \begin{cases} 0, & m \le 1 \\ n, & m > 1 \end{cases} \tag{23}$$

where, ${H^*}_m$ is homotopy polynomials and it is presented as

$${H^{1*}}_m = \frac{1}{m!}\frac{\partial^m \psi(x,t;q)}{\partial q^m}|_{q=0} \text{ and } \vec{u}(x,t;q) = \vec{u}_0 + q\vec{u}_1 + q^2\vec{u}_2 + \cdots \tag{24}$$

Utilizing Eqs. (21)-(22), one gets

$$u_m(x,t) = (\kappa_m + \hbar)u_{m-1}(x,t) - \left(1-\frac{\kappa_m}{n}\right)u_0(x,t) + \hbar \times \left({}_cR_\alpha\right)^{-1}\left[\left(\frac{\mathrm{u}}{s}\ {}_cR_\alpha[Au_{m-1}(\rho_i x,\sigma_i t) + {H^{1*}}_{m-1}(x,t) - \zeta(x,t)\right]\right)\right] \tag{25}$$

Step 4. It is then acquired via Cq-FHATM as

$$u(x,t) = \sum_{c=0}^{\infty} u_c(x,t) \tag{26}$$

IV. APPLICATIONS

This section describes how the approach is used to the conformable time-fractional Benjamin-Bona-Mahony equation (CTFBBME).

Application 4.1.

The nonlinear CTFBBME will be examined [18-19]**.**

$$D_t^{\alpha}u(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0, \;\; 0 < \alpha < 1, \tag{27}$$

with IC

$$\mathrm{u}(\mathrm{x},0) = e^x \tag{28}$$

Applying the CFFT to Eq. (27) and using IC, then it is found by

$$\frac{s}{u}\;{}_cR_{\alpha}[u(x,t)] - \frac{s}{u}u(x,0) + \;{}_cR_{\alpha}\left[u_x(x,t) + u(x,t)u_x(x,t\,) - u_{xxt}(x,t)\right] \;= 0. \tag{29}$$

Rewriting the Equation (29), then it is found by

$${}_cR_{\alpha}[u(x,t)] - u(x,0) + \frac{u}{s}\;{}_cR_{\alpha}\left[u_x(x,t) + u(x,t)u_x(x,t\,) - u_{xxt}(x,t)\right] \;= 0. \tag{30}$$

Via the homotopy analysis method, the nonlinear operator of $\varpi(x,t;q)$ is described by

$$N^1[\vec{u}(x,t;q)\,] = \;{}_cR_{\alpha}[\vec{u}(x,t;q)\,] - \vec{u}(x,t;q)\;(0^+) + \frac{u}{s}\;{}_cR_{\alpha}\left[\frac{\partial\vec{u}(x,t;q)}{\partial x}\right.$$
$$\left.+\vec{u}(x,t;q)\frac{\partial\vec{u}(x,t;q)}{\partial x} - \frac{\partial^3\vec{u}(x,t;q)}{\partial x^2\partial t}\right] \tag{31}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

Applying the suggested approach yields the following description of the $m-th$ order deformation equation

$${}_cR_{\alpha}[u_m(x,t) - \hbar_m u_{m-1}(x,t)] = \hbar\mathcal{R}_{1,m}(\vec{u}_{m-1}) \tag{32}$$

where,

$$\mathcal{R}_{1,m}(\vec{u}_{m-1}) = \;{}_cR_{\alpha}[u_{m-1}(x,t)] - \left(1 - \frac{\hbar_m}{n}\right)u_0(x,t)$$
$$+\frac{\mathrm{u}}{s}\;{}_cR_{\alpha}\left[\frac{\partial u_{m-1}(x,t)}{\partial x} + \sum_{r=0}^{m-1} u_r(x,t)\frac{\partial u_{m-1-r}(x,t)}{\partial x} - - \frac{\partial^3 u_{m-1}(x,t)}{\partial x^2\partial t}\right. \tag{33}$$

Eq. (34) is generated by applying the inverse CFFT (ICFFT) to Eq. (33).

$$u_m(x,t) = \hbar_m u_{m-1}(x,t) + \hbar\left(\;{}_cR_{\alpha}\right)^{-1}\left[\mathcal{R}_{1,m}(\vec{u}_{m-1})\right]. \tag{34}$$

Via IC, it is generated as

$$u_0(x,t) = e^x. \tag{35}$$

Putting the values of $m = 1, m = 2$ in the Eq. (34), Eqs. (36)-(37) are found by

$$u_1(x,t) = \frac{ht^{\alpha}}{\alpha}[e^x + e^{2x}], \tag{36}$$

$$u_2(x,t) = (n+\hbar)\frac{ht^{\alpha}}{\alpha}[e^{x}+e^{2x}] + h\left[h\frac{t^{2\alpha}}{2\alpha^{2}}[e^{x}+e^{2x}] + h\frac{t^{2\alpha}}{2\alpha^{2}}[e^{2x}+2e^{3x}]\right.$$
$$\left. + h\frac{t^{2\alpha}}{2\alpha^{2}}[e^{2x}+e^{3x}] - h\frac{(\alpha-1)t^{2\alpha}}{2\alpha^{2}}[e^{x}+4e^{2x}]\right] \quad (37)$$

As a result, equation (27)'s Cq-FHATM solution is determined to be

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^{m} \quad (38)$$

Two dimensional graphs for Cq-FHATM solution by distinct α values are shown in Figure 1.

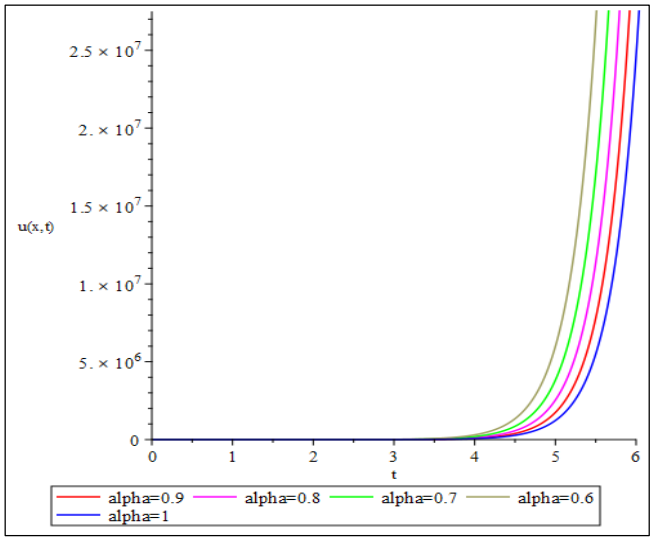
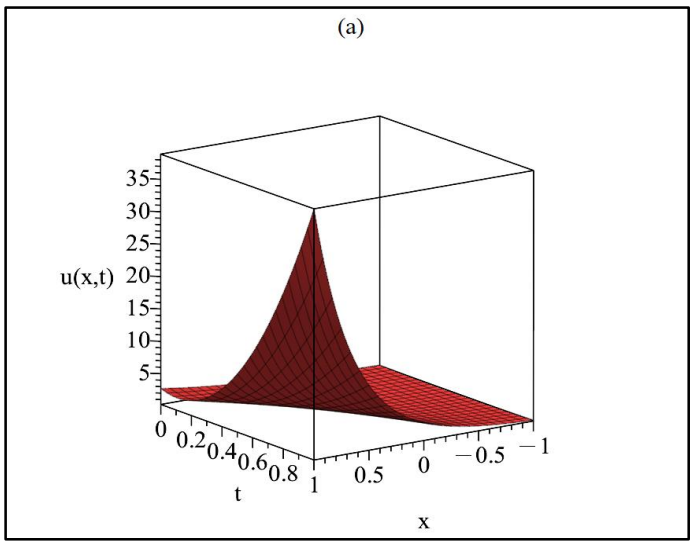
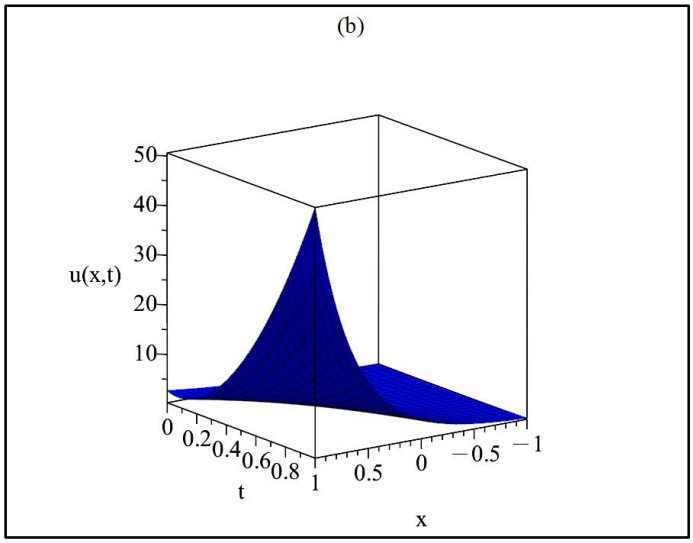
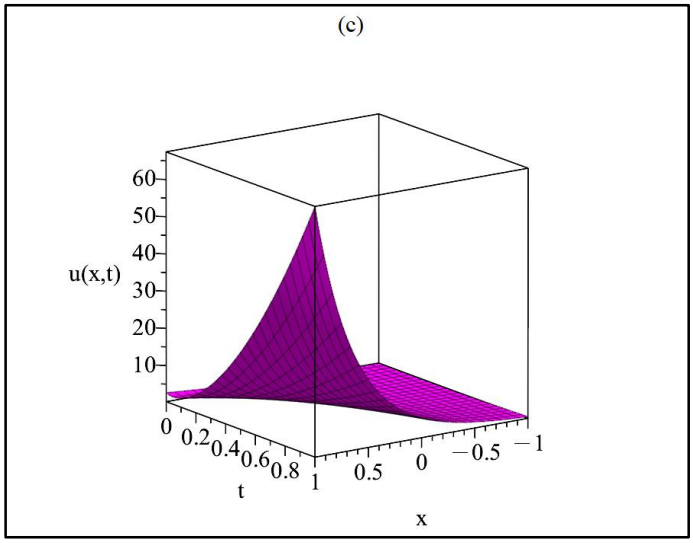


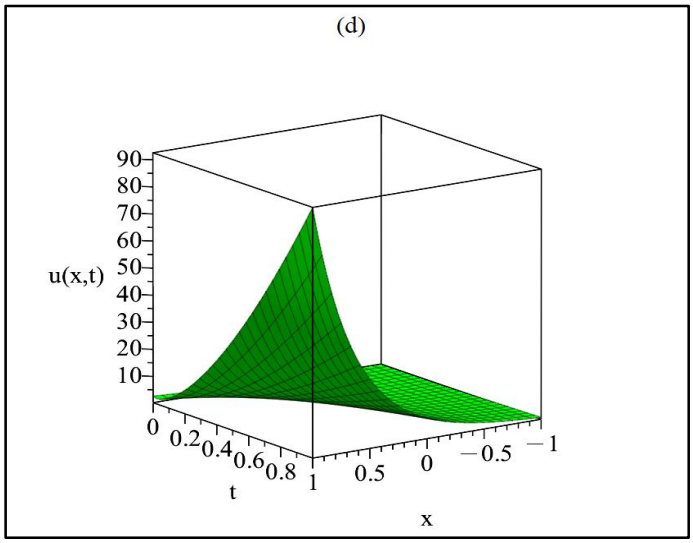
Figure 1. *2D plots of Cq-FHATM solution at* $n = 1$ *and* $\hbar = -1$*with distinct* α.

Figure 2 shows three dimensional graphs for the $u(x,t)$ solution of Cq-FHATM at the various values of α.




(b)
u(x,t)
50
40
30
20
10
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x


(c)
u(x,t)
60
50
40
30
20
10
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x


(d)
u(x,t)
90
80
70
60
50
40
30
20
10
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x

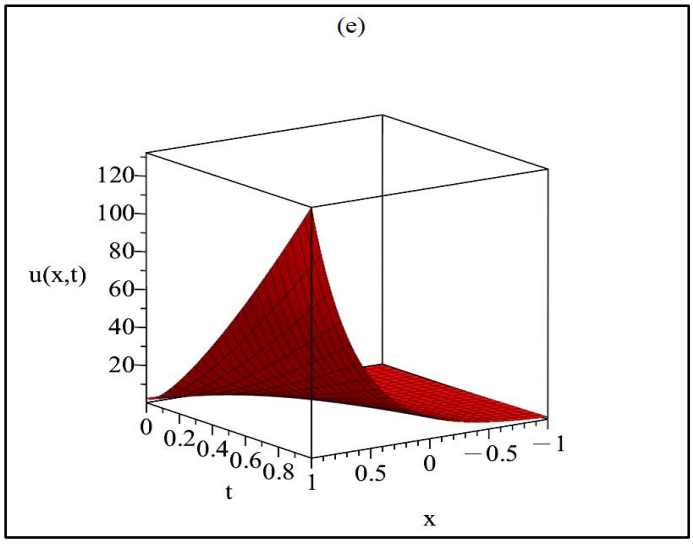


Figure 2. (a) *For* $\alpha = 1$, *Cq-FHATM solution* ***(b)*** *For* $\alpha = 0.9$, *Cq-FHATM solution* ***(c)*** *For* $\alpha = 0.8$, *Cq-FHATM solution* ***(d)*** *For* $\alpha = 0.7$, *Cq-FHATM solution* ***(e)*** *For* $\alpha = 0.6$, *Cq-FHATM solution for Eq. (27) at* $\hbar = -1, n = 1$.

The numerical values of Cq-FHATM solutions for CTFBBME for various fractional values α are shown in Table 1.

Table 1. *The numerical values of Cq-FHATM solutions at various fractional values* α.

x	t:	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
1	0.001	2.488083146	2.610169194	2.669209924	2.696109605	2.708220755
	0.002	2.398120368	2.549328682	2.634424167	2.677287310	2.698252214
	0.003	2.339627311	2.501448613	2.604262207	2.659753265	2.688376205
	0.004	2.299068286	2.461435618	2.577071329	2.643107676	2.678592725
	0.005	2.270810685	2.427071323	2.552104559	2.627163557	2.668901777
	0.006	2.251810001	2.397117206	2.528934038	2.611811300	2.659303360
	0.007	2.240178464	2.370785159	2.507281997	2.596978483	2.649797475
	0.008	2.234640058	2.347530699	2.486953274	2.582613468	2.640384120
	0.009	2.234279218	2.326955607	2.326955607	2.568677470	2.631063297
	0.010	2.238409564	2.308756228	2.449719482	2.555140189	2.621835005

Application 4.2.

The nonlinear CTFBBME will be examined [18-19].

$$D_t^{\alpha}u(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0, \;\; 0 < \alpha < 1 \tag{39}$$

with IC

$$\mathrm{u}(\mathrm{x},0) = x^2 \tag{40}$$

Applying the CFFT to Eq. (39) and using IC, then it is found by

$$\frac{s}{u}\;{}_cR_{\alpha}[u(x,t)] - \frac{s}{u}u(x,0) + \;{}_cR_{\alpha}\left[u_x(x,t) + u(x,t)u_x(x,t\,) - u_{xxt}(x,t)\right] \;= 0 \tag{41}$$

Rewriting the Equation (41), then it is found by

$${}_cR_{\alpha}[u(x,t)] - u(x,0) + \frac{u}{s}\;{}_cR_{\alpha}\left[u_x(x,t) + u(x,t)u_x(x,t\,) - u_{xxt}(x,t)\right] \;= 0 \tag{42}$$

Via the homotopy analysis method, the nonlinear operator of $\varpi(x,t;q)$ is described by

$$N^1[\vec{u}(x,t;q)\,] = \;{}_cR_{\alpha}[\vec{u}(x,t;q)\,] - \vec{u}(x,t;q)\;(0^+) + \frac{u}{s}\;{}_cR_{\alpha}\left[\frac{\partial\vec{u}(x,t;q)}{\partial x}\right.$$
$$\left.+\vec{u}(x,t;q)\frac{\partial\vec{u}(x,t;q)}{\partial x} - \frac{\partial^3\vec{u}(x,t;q)}{\partial x^2\partial t}\right] \tag{43}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

Applying the suggested approach yields the following description of the $m-th$ order deformation equation

$${}_cR_{\alpha}[u_m(x,t) - k_m u_{m-1}(x,t)] = \hbar\mathcal{R}_{1,m}(\vec{u}_{m-1}) \tag{44}$$

where,

$$\mathcal{R}_{1,m}(\vec{u}_{m-1}) = \;{}_cR_{\alpha}[u_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)u_0(x,t)$$
$$+\frac{\mathrm{u}}{s}\;{}_cR_{\alpha}\left[\frac{\partial u_{m-1}(x,t)}{\partial x} + \sum_{r=0}^{m-1} u_r(x,t)\frac{\partial u_{m-1-r}(x,t)}{\partial x} - - \frac{\partial^3 u_{m-1}(x,t)}{\partial x^2\partial t}\right. \tag{45}$$

Eq. (46) is generated by applying the inverse CFFT (ICFFT) to Eq. (45).

$$u_m(x,t) = k_m u_{m-1}(x,t) + \hbar\left(\;{}_cR_{\alpha}\right)^{-1}\left[\mathcal{R}_{1,m}(\vec{u}_{m-1})\right] \tag{46}$$

Via IC, it is generated as

$$u_0(x,t) = x^2 \tag{47}$$

Putting the values of $m = 1, m = 2$ in the Eq. (46), Eqs.(48)-(49) are found by

$$u_1(x,t) = \frac{ht^{\alpha}}{\alpha}[2x + 2x^3], \tag{48}$$

$$u_2(x,t) = (n+\hbar)\frac{ht^{\alpha}}{\alpha}[2x + 2x^3] + h\left[h\frac{t^{2\alpha}}{2\alpha^2}[2 + 6x^2] + hx^2\frac{t^{2\alpha}}{2\alpha^2}[2 + 6x^2]\right.$$
$$\left. +2\,h\frac{t^{2\alpha}}{2\alpha^2}[2x + 2x^3] - 12hx\frac{(\alpha-1)t^{2\alpha}}{2\alpha^2}\right] \tag{49}$$

As a result, equation (39)'s Cq-FHATM solution is determined to be

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m \tag{50}$$

Two dimensional graphs for Cq-FHATM solution by distinct μ values are shown in Figure 3.

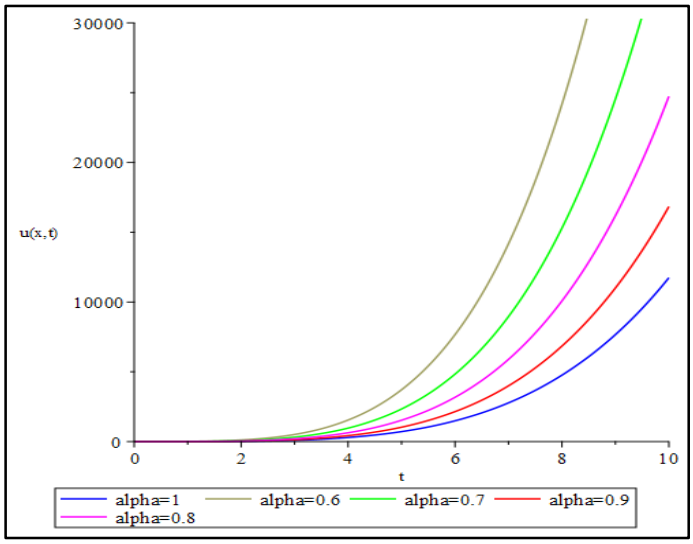
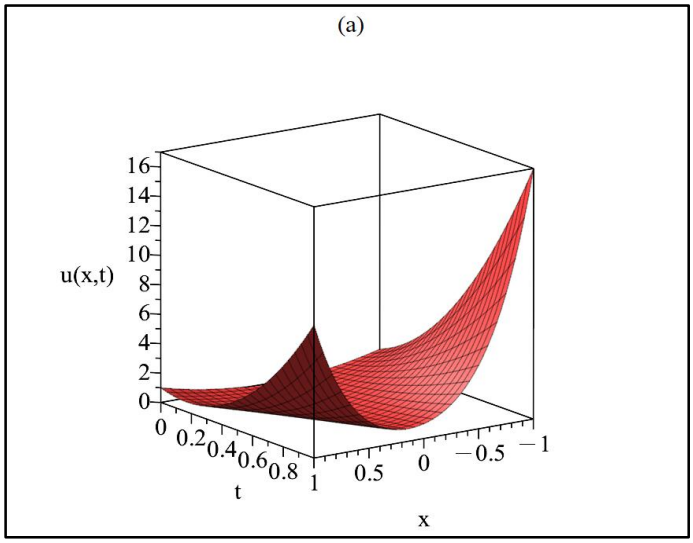
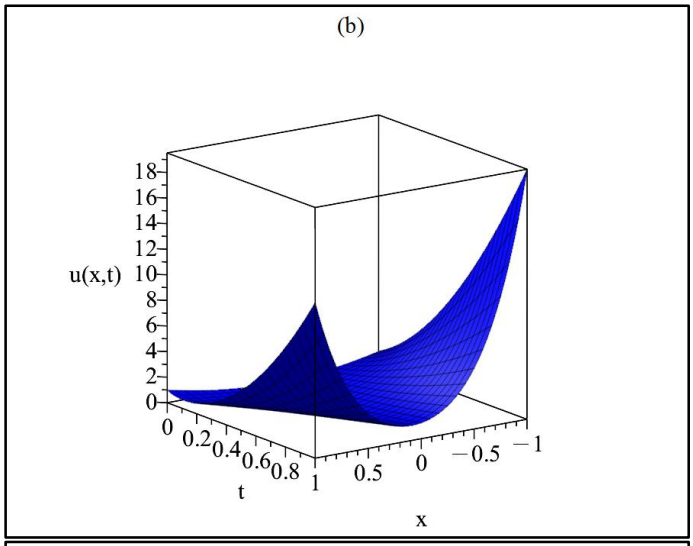
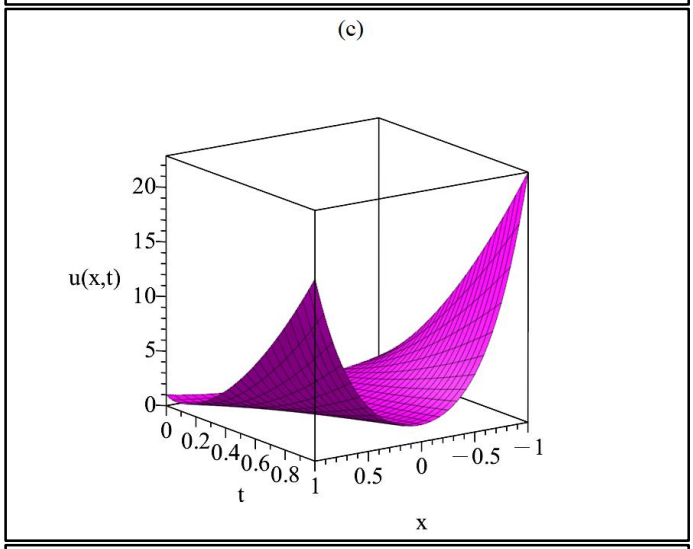


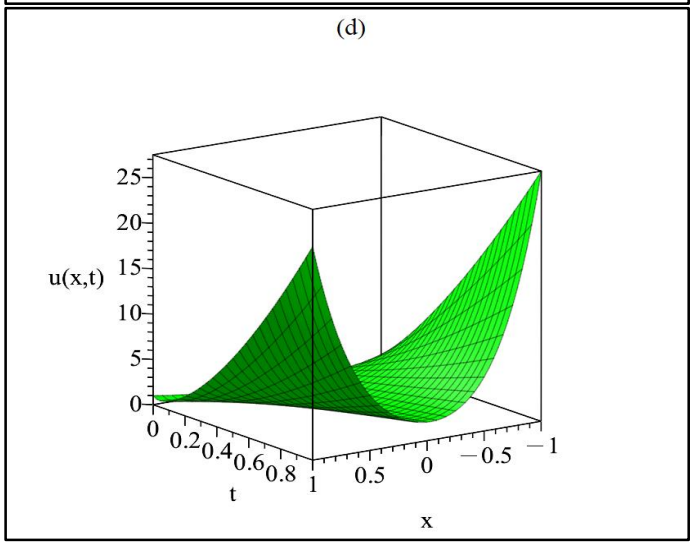
Figure 3. *2D plots of Cq-FHATM solution at* $\hbar = -1, n = 1$ *with distinct* α.

Three dimensional graphs for Cq-FHATM solution by distinct α values are shown in Figure 4.




(b)
u(x,t)
18
16
14
12
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x


(c)
u(x,t)
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x


(d)
u(x,t)
25
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1
t
0.5
0
−0.5
−1
x

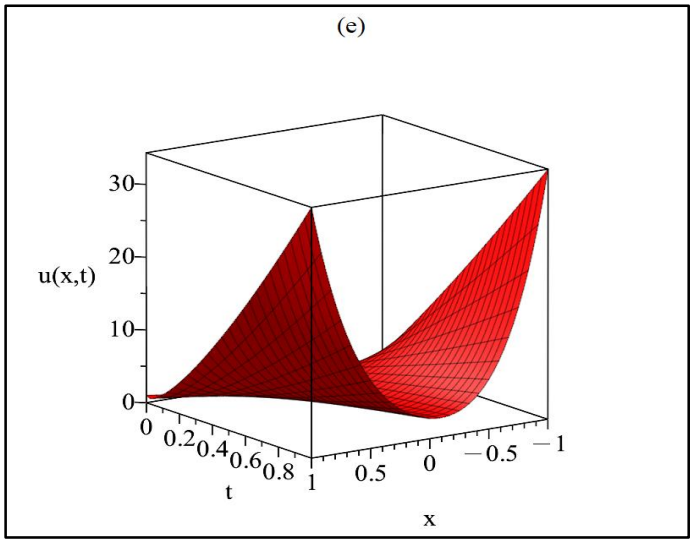


Figure 4. (a) *For* $\alpha = 1$, *Cq-FHATM solution* ***(b)*** *For* $\alpha = 0.9$, *Cq-FHATM solution* ***(c)*** *For* $\alpha = 0.8$, *Cq-FHATM solution* ***(d)*** *For* $\alpha = 0.7$, *Cq-FHATM solution* ***(e)*** *For* $\alpha = 0.6$, *Cq-FHATM solution for Eq. (39) at* $\hbar = -1, n = 1$.

The numerical values of Cq-FHATM solutions for CTFBBME for various fractional values α are shown in Table 2.

Table 2. *The numerical values of Cq-FHATM solutions at various fractional values* α.

x	t:	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
1	0.001	0.904387999	0.956386797	0.980421525	0.991194095	0.996012000
	0.002	0.862933274	0.930952910	0.966333685	0.983667684	0.992048000
	0.003	0.833290478	0.910335797	0.953959204	0.976611779	0.988108000
	0.004	0.810289370	0.892590204	0.942662167	0.969871333	0.984192000
	0.005	0.791796748	0.876877387	0.932157857	0.963374400	0.980300000
	0.006	0.776667404	0.862732924	0.922284504	0.957079109	0.976432000
	0.007	0.764195379	0.849863419	0.912937843	0.950957970	0.972588000
	0.008	0.753905942	0.838067476	0.904045103	0.944991508	0.968768000
	0.009	0.745460057	0.827198673	0.895552619	0.939165177	0.964972000
	0.010	0.738604638	0.817145955	0.887419173	0.933467681	0.961200000

V. RESULTS AND DISCUSSION

Figure 1 illustrates the two-dimensional graphs of the $u(x,t)$ obtained by Cq-FHATM for several values of α, namely $\alpha = 0.6,\ \alpha = 0.7,\ \alpha = 0.8,\ \alpha = 0.9,$ and $\alpha = 1$ in the context of CTFBBME. Figure 2 displays three-dimensional graphs, denoted as $u(x,t)$, obtained through the use of Cq-FHATM for several values of α, specifically $\alpha = 0.6,\ \alpha = 0.7,\ \alpha = 0.8,\ \alpha = 0.9,$ and $\alpha = 1$. The $u(x,t)$ solution produced by Cq-FHATM for various values of $\alpha = 0.6,\ \alpha = 0.7,\ \alpha = 0.8,\ \alpha = 0.9,$ and $\alpha = 1$ for Eq. (27) is presented in Table 1. The two-dimensional graphs of the results achieved for the CTFBBME, as shown in Example 4.2, may be observed in Figure 3. The values of α were varied to obtain these outcomes. The 2D graphs of the solutions obtained for different α values in Example 4.2 are shown in Figure 3. The Figure 4 displays the 3D graphs representing the outcomes produced for various α values in Example 4.2. The $u(x,t)$ solution produced by Cq-FHATM for various values of $\alpha = 0.6,\ \alpha = 0.7,$ $\alpha = 0.8,\ \alpha = 0.9,$ and $\alpha = 1$ for Eq. (39) is presented in Table 2.

VI. CONCLUSION

In this paper, the Cq-FHATM, a new hybrid method consisting of the unification the conformable formable integral transform and the q-homotopy analysis transform method, is introduced. CFBBM equations have been examined by conformable q-formable analysis transform method. The numerical solutions of the CFBBM equation were obtained for the first time and contributed to the literature. Figure 1 shows that as α converges 1, the solutions approach exact solutions and the behaviors of the solutions is consistent. Figure 2 illustrates that as α converges 1, the solutions get more analytical, and the behavior of the three-dimensional solutions remains consistent. For Example 4.2, Figure 3 illustrates that as α converges 1, the solutions approach analytical solutions, and their behavior is constant. Figure 4 illustrates that as α converges 1, the solutions get more analytical, and the behavior of the three-dimensional solutions remains consistent in Example 4.2. The method's capacity to decompose intricate equations into more manageable components, in conjunction with its precision and alignment with other numerical approaches, renders it a useful asset in the realm of mathematics. Additionally, the Cq-FHATM presents numerous benefits in comparison to conventional numerical methods, rendering its implementation more straightforward and yielding enhanced precision. Due to its considerable potential, it is anticipated that this approach will assume a progressively significant role in addressing nonlinear differential equations in the forthcoming years. As a result, it is possible to conclude that the used approach is extraordinarily systematic and successful in assessing partial differential equations with fractional order in a systematic and improved manner. This method is recommended for the analysis of conformable FPDEs which is used science and technology. Cq-FHATM does not require discretization or linearization, preserving the original structure of the equation and reducing computational complexity. Furthermore, it provides adjustable convergence parameters, which allow for precise control over the solution process and ensure higher accuracy. Compared to Adomian decomposition or variational iteration methods, Cq-FHATM demonstrates superior convergence rates and a broader applicability to nonlinear problems. However, the method also has certain limitations. Its reliance on parameter tuning can introduce variability and subjectivity in achieving optimal results, which may complicate reproducibility. Additionally, while it simplifies fractional calculus, the computational cost can grow significantly for high-dimensional or large-scale problems. In contrast, traditional numerical methods, though computationally intensive, are often easier to implement for well-studied standard equations. Finally, the theoretical framework of Cq-FHATM, particularly for complex and multi-dimensional systems, requires further validation and development to match the robustness of more established techniques.

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